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## Some Remarks on Solutions of Fermat's Last Equation in Terms of Wright's Hypergeometric Function

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# SOME REMARKS ON SOLUTIONS OF FERMAT'S LAST EQUATION IN TERMS OF WRIGHT'S HYPERGEOMETRIC FUNCTION

## INTRODUCTION

In the seventeenth century the French mathematician and lawyer, Pierre de Fermat (1601-1665) stated that he had a proof of a result which is now known as

**Fermat's Last Theorem:** If  $n$  is an integer greater than 2, then the equation

$$x^n + y^n = z^n \quad (1)$$

has no solution in positive integers.

Fermat did not give a proof of this result and, to this day, no one else has either.

Recently it has been shown [1,2] that if Eq. (1) has a solution in positive real numbers with  $x < y$ , then the exponent in Eq. (1) is given by

$$n = \log_{y/z} [\lambda \Psi(\lambda)] , \quad (2)$$

where

$$\lambda = \log_{x/z}(y/z) , \quad (3)$$

and

$$\Psi(\lambda) = {}_1\Psi_1 \left[ \begin{matrix} (\lambda, \lambda) & ; & -1 \\ (\lambda + 1, \lambda - 1) & ; & \end{matrix} \right] , \quad 0 < \lambda < 1 . \quad (4)$$

The function  ${}_1\Psi_1$  in Eq. (4) is a special case of Wright's generalized hypergeometric function  ${}_p\Psi_q$  whose series representation is given by

$${}_p\Psi_q \left[ \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) & ; & z \\ (\beta_1, B_1), \dots, (\beta_q, B_q) & ; & \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(\alpha_i + A_i n)}{\prod_{i=1}^q \Gamma(\beta_i + B_i n)} \frac{z^n}{n!} . \quad (5)$$

In this paper we shall give an equivalent form of Fermat's last theorem in terms of the Wright function  $\Psi(\lambda)$  and the related Wright function  $\hat{\Psi}(\lambda)$  defined by

$$\hat{\Psi}(\lambda) \equiv {}_1\Psi_1 \left[ \begin{matrix} (1, \lambda) \\ (2, \lambda - 1) \end{matrix} ; -1 \right], \quad 0 < \lambda < 1. \quad (6)$$

In addition, we shall give some elementary properties of the functions  $\Psi(\lambda)$  and  $\hat{\Psi}(\lambda)$ .

## MAIN RESULT

**Theorem:** The following are equivalent:

- (a) Fermat's Last Theorem is false.
- (b) There is a transcendental number  $\lambda \in (0, 1)$  and a positive integer  $n > 2$  such that  $\lambda\Psi(\lambda)$  and  $\hat{\Psi}(\lambda)$  are simultaneously  $n$ -th powers of rational numbers in  $(0, 1)$ .

In order to prove this theorem we prove the following two lemmas.

**Lemma 1:** If Eq. (1) has a solution in positive integers with  $x < y$  and  $\gcd(x, y, z) = 1$ , then there is a unique transcendental number  $\lambda$  in the interval  $(0, 1)$  such that

$$\left(\frac{x}{z}\right)^\lambda = \frac{y}{z}. \quad (7)$$

**Proof.** The function  $f(\lambda) \equiv (x/z)^\lambda$  is a decreasing continuous function of  $\lambda$  on  $[0, 1]$ , since  $x/z < 1$ . Moreover,  $f(1) = x/z < y/z < 1 = f(0)$ , so by the

intermediate value theorem there is a unique  $\lambda$  in the interval  $(0,1)$  such that

$$f(\lambda) = \left(\frac{x}{z}\right)^\lambda = \frac{y}{z}.$$

Since  $\gcd(x,y,z) = 1$ , the integers  $x,y,z$  are pairwise relatively prime. Suppose  $\lambda = a/b$  is rational, where  $a$  and  $b$  are relatively prime positive integers. Then by Eq. (7),

$$\left(\frac{x}{z}\right)^{a/b} = \frac{y}{z}$$

or

$$x^a z^b = y^b z^a. \quad (8)$$

Since  $x,y,z$  are pairwise relatively prime, there must be a prime  $p$  that divides  $y$  but divides neither  $x$  nor  $z$ . So  $p$  divides the right side of Eq. (8) but not the left side, and we have a contradiction. Thus  $\lambda$  must be irrational.

This corrects the proof that  $\lambda$  is irrational given in [1, p.7]. In [1,2] a second proof of this result is given.

In order to show the transcendence of  $\lambda$  we shall need the result of Gelfond and Schneider [3, p. 21] which states that if  $\alpha$  and  $\beta$  are algebraic,  $\alpha \neq 0, 1$  and  $\beta$  is irrational, then  $\alpha^\beta$  is transcendental.

From Eq. (7), since  $x/z$  is algebraic,  $0 < x/z < 1$ , and  $\lambda$  is irrational, then if  $\lambda$  is also algebraic,  $y/z$  would be transcendental, which it is not. Hence  $\lambda$  must be transcendental and the lemma is proved. Note that Eq. (7) is just another form of Eq. (3).

**Lemma 2:** For all  $\lambda \in (0,1)$  we have the identity

$$\lambda \Psi(\lambda) + \hat{\Psi}(\lambda) = 1. \quad (9)$$

**Proof.** Using Eq. (5) we see that since  $\Gamma(1+z) = z\Gamma(z)$ ,

$$\begin{aligned}
 {}_1\Psi_1 \left[ \begin{matrix} (1, \lambda) & ; & -1 \\ (2, \lambda-1) & ; & \end{matrix} \right] &= \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(2+(\lambda-1)n)} \frac{(-1)^n}{n!} \\
 &= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(2+(\lambda-1)n)} \frac{(-1)^n}{\Gamma(1+n)} \\
 &= 1 + \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda+\lambda n)}{\Gamma(\lambda+1+(\lambda-1)n)} \frac{(-1)^{n+1}}{\Gamma(2+n)} \\
 &= 1 - \sum_{n=0}^{\infty} \frac{\lambda(1+n)\Gamma(\lambda+\lambda n)}{\Gamma(\lambda+1+(\lambda-1)n)} \frac{(-1)^n}{(1+n)\Gamma(1+n)} \\
 &= 1 - \lambda \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+\lambda n)}{\Gamma(\lambda+1+(\lambda-1)n)} \frac{(-1)^n}{n!} .
 \end{aligned}$$

Now using Eqs. (4)–(6) we have the result Eq. (9).

**Proof of Theorem:** If Fermat's Theorem is false, then Eq. (1) holds for some  $n > 2$  and positive integers  $x < y < z$ . By Lemma 1 there is a unique transcendental number  $\lambda \in (0, 1)$  such that Eq. (2) holds, i.e.,

$$\lambda\Psi(\lambda) = (y/z)^n$$

and from Lemma 2,

$$\begin{aligned}
 \hat{\Psi}(\lambda) &= 1 - \lambda\Psi(\lambda) \\
 &= 1 - (y/z)^n
 \end{aligned}$$

so that

$$\hat{\Psi}(\lambda) = (x/z)^n . \tag{10}$$

(We shall derive Eq. (10) in another way in the next section.)

Conversely, if for some transcendental  $\lambda \in (0, 1)$ , both  $\lambda\Psi(\lambda)$  and  $\hat{\Psi}(\lambda)$  are  $n$ -th ( $n > 2$ ) powers of some rational numbers, say

$$\lambda\Psi(\lambda) = (u_1/v_1)^n, \quad \hat{\Psi}(\lambda) = (u_2/v_2)^n,$$

then from Lemma 2 we have

$$(u_1v_2)^n + (u_2v_1)^n = (v_1v_2)^n$$

and the theorem is proved.

#### ANOTHER DERIVATION OF EQ. (10)

From Eq. (3) we have  $y = x^\lambda z^{1-\lambda}$  which when substituted into Eq. (1) gives

$$(x/z)^n + (x/z)^{\lambda n} - 1 = 0.$$

Setting  $\xi = (x/z)^n$  we have

$$\xi + \xi^\lambda - 1 = 0, \quad 0 < \lambda < 1. \quad (11)$$

In order to solve Eq. (11) for  $\xi$  as a function of  $\lambda$  we shall need the following result. The positive root of the trinomial equation

$$\xi^p + \mu\xi^q - 1 = 0, \quad p > q > 0 \quad (12)$$

is given by

$$\xi = \frac{1}{p} {}_1\Psi_1 \left[ \begin{matrix} (1/p, q/p) \\ (1 + 1/p, q/p - 1) \end{matrix} ; -\mu \right], \quad (13)$$

for real  $\mu$  such that

$$|\mu| < (q/p)^{-q/p}(1 - q/p)^{q/p-1} \leq 2. \quad (14)$$

For integers  $p$  and  $q$ , Mellin in 1915 [4] gave the series solution, Eq. (13), of the trinomial Eq. (12). However, his result is valid for real  $p$  and  $q$ . Lagrange circa 1768 [5] derived essentially the same result in terms of a series of binomial coefficients. In 1758 [6] Lambert gave the solution of trinomial equations. Ramanujan [7, pp. 71, 307] studied and derived solutions of trinomials in Chapter 3 of his notebooks (1903-1914) and in his first quarterly report (1913).

Now set  $p = 1$ ,  $q = \lambda$ ,  $\mu = 1$  in Eqs. (12)-(13). Then the condition Eq. (14) is easily checked and using the definition of  $\hat{\Psi}(\lambda)$  given by Eq. (6) we arrive at Eq. (10).

#### ADDITIONAL OBSERVATIONS

From Eq. (10) we see that the exponent in Eq. (1) is also given by

$$n = \log_{x/z} \hat{\Psi}(\lambda). \quad (15)$$

This result is somewhat less complex than the result Eq. (2). We may also write

$$\begin{aligned} \hat{\Psi}(\lambda) &= \lambda \sum_{n=0}^{\infty} (-1)^n {}_2F_1[-n, (1-\lambda)(n+2); 2; 1], \\ \hat{\Psi}(\lambda) &= \lambda \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \binom{\lambda(2+n)-1}{n}, \quad 0 < \lambda < 1. \end{aligned}$$

The latter equations follow directly from Eq. (9) and [1, Eqs. (19)-(20)]. In addition from Eqs. (2), (3), (9), and (15) we observe that

$$[\hat{\Psi}(\lambda)]^\lambda = 1 - \hat{\Psi}(\lambda) ,$$

$$[1 - \lambda\Psi(\lambda)]^\lambda = \lambda\Psi(\lambda) .$$

## CONCLUSION AND ACKNOWLEDGEMENT

Fermat's Last Theorem may be stated in equivalent form using Wright's generalized hypergeometric function  ${}_1\Psi_1$ .

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